Abstract

This paper presents a lower bound of \(\tilde{\Theta}(D + \sqrt{n})\) on the time required for the distributed construction of a minimum-weight spanning tree (MST) in \(n\)-vertex networks of diameter \(D = \Omega(\log n)\), in the bounded message model. This establishes the asymptotic near-optimality of existing time-efficient distributed algorithms for the problem, whose complexity is \(O(D + \sqrt{n} \log^* n)\).

1 Introduction

The study of distributed algorithms for MST construction was initiated by the pioneering work of Gallager et al. [GHS83], which introduced a basic distributed technique for the problem and presented a message-optimal algorithm with time complexity \(O(n \log n)\) on an \(n\)-vertex network. This result was later improved to a message-optimal algorithm with time complexity \(O(n)\) by Awerbuch [A87].

However, for many natural distributed network problems, the parameter controlling the time complexity is not the number of vertices but rather the network’s diameter \(D\), namely, the maximum distance between any two vertices (measured in hops). This holds, for example, for leader election and related problems [P90].

It is easy to verify that \(\Omega(D)\) time is required for distributed MST construction in the worst case. More formally, for every two integers \(n \geq 2\) and \(1 \leq D \leq \lceil n/2 \rceil\) there exist \(n\)-vertex networks of diameter \(D\) (say, based on a 2\(D\)-vertex ring with \(n - 2D\) vertices attached to it as leaves) on which any distributed MST algorithm will require at least \(D\) time.

Hence a natural question is whether \(O(D)\)-time algorithms exist for distributed MST construction as well. More generally, the problem of devising faster-than-\(O(n)\) (though possibly not message-optimal) distributed algorithms for MST construction was introduced in [GKP98].

Clearly, in the extreme model allowing the transmission of an unbounded-size message on a link in a single time unit (cf. [L87]), the problem can be trivially solved in time \(O(D)\) by collecting the entire graph’s topology and all the edge weights into a central vertex, computing an MST locally and broadcasting the result throughout the network. The problem thus becomes interesting in the more realistic, and rather common, \(B\)-bounded-message model (henceforth referred to simply as the \(B\) model), in which message size is bounded by some value \(B\) (usually taken to be either constant or \(O(\log n)\)), and a vertex may send at most one message on each edge at each time unit.

The algorithm presented in [GKP98] for distributed MST construction in this model (with \(B = O(\log n)\)-bit messages) has time complexity \(O(D + n^{1/3} \log^* n)\) for \(\epsilon = \ln 6 / \ln 3 \approx 0.613\). This was later improved to \(O(D + \sqrt{n} \log^* n)\) in [KP98]. Similar bounds were recently obtained by us using other algorithmic methods, but none of those methods were able to break the \(\sqrt{n}\)-time barrier, indicating that distributed MST might be harder than other distributed network problems such as leader election or BFS tree construction.

The current paper concerns establishing the asymptotic near-optimality of the algorithm of [KP98], by showing that \(\tilde{\Theta}(\sqrt{n})\) is a lower bound\(^1\) as well, even on low diameter networks. Specifically, for any integers \(K, m \geq 2\), we construct a family of \(O(m^2 K)\)-vertex networks of diameter \(D = O(K m)\) for

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\(^1\)\(\tilde{\Theta}\) is a relaxed variant of the \(\Omega\) notation that ignores polylog factors.

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which \( \Omega(m^K/BK) \) time is required for constructing a minimum spanning tree in the \( B \) model. Fixing some positive integer \( m \geq 2 \), we get that for every integer \( n \geq 1 \) there exists a family of \( n \)-vertex networks of diameter \( \Theta(\log n) \) for which MST construction requires \( \Omega(\sqrt{n}/B \log n) \) time in the \( B \) model.

While it is not clear that the \( \Theta(\log n) \) limitation on the diameters for which the lower bound holds is essential, some limitation must apparently exist. This follows from the observation that the \( n \)-vertex complete graph \( (D = 1) \) admits a simple \( O(\log n) \) time distributed MST construction algorithm.

Towards proving the lower bound on distributed MST construction, we first establish a lower bound on the time complexity of a problem referred to as the mailing problem, which can be informally stated as follows. Given a particular type of graph named \( F^K_m \), for integers \( m, K \geq 2 \), and two vertices \( s \) and \( r \) in it, it is required to deliver a \( m^K \)-bit string \( \mathcal{X} \) generated in \( s \) to \( r \). The graph \( F^K_m \) has \( n = O(m^2K) \) vertices and diameter \( O(Km) \), yet we need to show that the time required for mailing from \( s \) to \( r \) on \( F^K_m \) in the \( B \) model is considerably larger, namely, \( \Omega(m^K/BK) = \Omega(\sqrt{n}/BK) \).

The rest of the paper is organized as follows. First, a definition of the model is given in Section 2. Section 3 handles the mailing problem for the case of \( K = 2 \). Specifically, it introduces the mailing problem formally, defines the graphs \( F^2_m \) and shows a lower bound on time complexity of the mailing problem for \( m^2 \)-bit strings on \( F^2_m \). This result is then used in Section 4 to prove that the same lower bound applies also to the time complexity of the MST problem on weighted versions of the graphs \( F^2_m \). Section 5 generalizes these results to the graphs \( F^K_m \) for \( K \geq 3 \), thus establishing the desired result.

2 The model

A point-to-point communication network is modeled as an undirected graph \( G(V,E) \), where the vertices of \( V \) represent the network processors and the edges of \( E \) represent the communication links connecting them. Vertices are allowed to have unique identifiers. The vertices do not know the topology or the edge weights of the entire network, but they may know the ID’s of their neighbors and the weights of the corresponding edges.

A weight function \( \omega : E \rightarrow \mathbb{R}^+ \) associated with the graph assigns a real nonnegative weight \( \omega(e) \) to each edge \( e = (u,v) \in E \). The weight \( \omega(e) \) is known to the adjacent vertices, \( u \) and \( v \). The vertices can communicate only by sending and receiving messages over the communication links. Communication is carried out in a synchronous manner, i.e., all the vertices are driven by a global clock. Messages are sent at the beginning of each round, and are received at the end of the round. (Clearly, our lower bounds hold for asynchronous networks as well.) At most one \( B \)-bit message can be sent on each link in one direction on every round. The model also allows vertices to detect the absence of a message on a link at a given round, which can be used to convey information. Hence at each communication round, a link can be at one of \( 2^B + 1 \) possible states, i.e., it can either transmit any of \( 2^B \) possible messages, or remain silent.

The length of a path \( p \) in the network is the number of edges it contains. The distance between two vertices \( u \) and \( v \) is defined as the length of the shortest path connecting them in \( G \). The diameter of \( G \), denoted \( D \), is the maximum distance between any two vertices of \( G \).

3 A lower bound for the mailing problem on \( F^2_m \)

This section introduces the mailing problem formally, defines the graphs \( F^2_m \) and establishes a lower bound on time complexity of the mailing problem for \( m^2 \)-bit strings on \( F^2_m \).

3.1 The mailing problem

Consider the following situation. We are given some graph \( G \) with two distinguished vertices denoted \( s \) and \( r \), referred to as the sender and the receiver, respectively. Both the sender \( s \) and the receiver \( r \) store \( b \) boolean variables each, \( X^s_1, \ldots, X^s_b \) and \( X^r_1, \ldots, X^r_b \) respectively, for some integer \( b \geq 1 \). An input of the problem consists of an initial assignment \( \mathcal{X} = \{x_i \mid 1 \leq i \leq b\} \), where \( x_i \in \{0,1\} \), to the variables of \( s \), such that \( X^s_i = x_i \). Given such an instance, the mailing problem requires \( s \) to deliver the string \( \mathcal{X} \) to the receiver \( r \), i.e., upon termination, the variables of \( r \) should contain the output \( X^r_i = x_i \) for every \( 1 \leq i \leq b \). Henceforth, we refer to this problem as \( \text{Mail}(G,s,r,b) \). Throughout the remainder of this section we consider this problem on graphs \( F^2_m \) with \( b = m^2 \) for some integer \( m \geq 2 \). In Section 5 we deal with the problem on graphs \( F^K_m \) with \( b = m^K \) for \( K \geq 3 \).

3.2 The graphs \( F^2_m \)

Let us now define the collection of graphs denoted \( F^2_m \), for \( m \geq 2 \). The two basic units in the construction

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are the ordinary path $P$ on $m^2 + 1$ vertices,

$$V(P) = \{v_0, \ldots, v_{m^2}\}$$

and

$$E(P) = \{e_i = (v_i, v_{i+1}) | 0 \leq i \leq m^2 - 1\},$$

and the highway $H$ on $m + 1$ vertices,

$$V(H) = \{h_i | 0 \leq i \leq m\}$$

and

$$E(H) = \{f_i = (h_i, h_{i+1}) | 0 \leq i < m \}. $$

Each highway vertex $h_i$ is connected to the corresponding path vertex $v_i$ by a spoke edge $(h_i, v_i)$, as in Fig. 1.

The graph $F^2_m$ is constructed by taking $m^2$ copies of the ordinary path $P$, denoted $P^1, \ldots, P^{m^2}$, and connecting all of them to the same highway $H$. The vertex $h_0$ is the intended sender $s$, and the vertex $h_{m^2}$ is the intended receiver $r$. (See Fig. 2.)

![Figure 1](image1.png) \hspace{1cm} ![Figure 2](image2.png)

Figure 1: The connections between the path and the highway.

Figure 2: The graph $F^2_m$.

Visualizing the graph $F^2_m$ as organized in a cylindrical shape, the spoke edges can be grouped into $m + 1$ stars $S_i$, $0 \leq i \leq m$, where each star $S_i$ consists of the highway vertex $h_i$ and the $m^2$ vertices $v_{i1}^1, \ldots, v_{i1}^{m^2}$ connected to it by spoke edges. Hence

$$V(S_i) = \{h_i \} \cup \{v_{i1}^1, \ldots, v_{i1}^{m^2}\}$$

and

$$E(S_i) = \{(v_{i1}^j, h_i) | 1 \leq j \leq m^2\}.$$ The vertex and edge sets of the graph $F^2_m$ are thus

$$V(F^2_m) = V(H) \cup \bigcup_{j=1}^{m^2} V(P^j)$$

and

$$E(F^2_m) = \bigcup_{i=0}^{m} E(S_i) \cup \bigcup_{j=1}^{m^2} E(P^j) \cup E(H).$$

Fact 3.1. The graph $F^2_m$ consists of $n = \Theta(m^4)$ vertices, and its diameter is $O(m)$.

3.3 The lower bound

We would now like to prove that solving the mailing problem on the graph $F^2_m$ with a $b = m^2$-bit string $X$ requires $\Omega(m^2/B)$ time in the $B$ model. Intuitively, this happens because routing the string $X$ from $s$ to $r$ along ordinary paths would be too slow, hence our only hope is to route the string along the highway, or at least use interleaved paths, mixing highway segments with segments of ordinary paths. However, $F^2_m$ does not have sufficient capacity for routing all $m^2$ bits from $s$ to $r$ along such short (or "relatively short") paths.

This intuition yields a rather simple proof of the claim if we limit ourselves to a restricted class of algorithms, referred to as explicit delivery algorithms. These are algorithms in which the input bits are required to be delivered in an explicit way, namely, each bit $x_i$ must be shipped from $s$ to $t$ along some path $p_i$ (naturally, the paths of different bits may be identical, or partly overlap). However, we would like the
lower bound to apply also to arbitrary algorithms, in which the information can be conveyed from \( s \) to \( r \) in arbitrary ways. This may include applying arbitrary functions to the bits at \( s \) and sending the resulting values, possibly modifying and “recombining” these values in intermediate nodes along the way, in a way that will allow \( r \) to extract the original bits from the messages it receives. For handling such a general class of algorithms, the proof must be formalized in a more careful way.

Let us start with an outline of the proof. Consider the set of possible states a vertex \( v \) may be in at any given stage \( t \) of the execution of a mailing algorithm on some \( m^2 \)-bit input \( X \). (The state of a vertex consists of all its local data, hence it is affected by its input, topological knowledge, and history, namely, all incoming messages.) As the computation progresses, the tree of possible executions diverges, and thus the set of possible states of \( v \) becomes larger. In particular, when the execution starts at round 0, each of the vertices is in one specific initial local state, except for the sender \( s \), which may be in any one of \( 2^{m^2} \) states, determined by the value of the input string \( X \). Upon termination, the string \( X \) should be known to the receiver \( r \), meaning that \( r \) should be in one of \( 2^{m^2} \) states. Our argument is based on analyzing the growth process of the sets of possible states, and showing that this process is slow, forcing the algorithm to spend at least \( \Omega(m^2/B) \) time until the set of possible states of \( r \) is of size \( 2^{m^2} \).

Let us continue with a more detailed and formal proof. Consider some arbitrary algorithm \( A_{maid} \) and let \( v^X \), \( \phi^X \) denote the execution of \( A_{maid} \) on an \( m^2 \)-bit input \( X \) in the graph \( F_m^2 \). For \( 1 \leq i \leq m \), define the tail set of the graph \( F_m^2 \), denoted \( T_i \), as follows:

\[
T_i(P_j) = \{ v_i^j \mid i \leq l \leq m^2 \}.
\]

Let \( \beta(i) \) denote the least integer \( \delta \) such that \( \delta m \geq i \), and define the tail of \( H \) as

\[
T_i(H) = \{ h_{jm} \mid \beta(i) \leq j \leq m \}.
\]

Now, the tail set of \( F_m^2 \) is the union of those tails,

\[
T_i = T_i(H) \cup \bigcup_j T_i(P_j).
\]

(See Fig. 3.) For \( i = 0 \), the definition is slightly different, letting

\[
T_0 = V \setminus \{ h_0 \}.
\]

Denote the state of the vertex \( v \) at the beginning of round \( t \) during the execution \( \phi^X \) on the input \( X \) by \( \sigma(v,t,X) \). In two different executions \( \phi^X \) and \( \phi^{X'} \), a vertex reaches the same state \( \sigma \) at time \( t \), i.e., \( \sigma(v,t,X) = \sigma(v,t,X') \), if it receives the same sequence of messages on each of its incoming links; for different sequences, the states are distinguishable.

For a given set of vertices \( U = \{v_1, \ldots, v_l\} \subseteq V \), a configuration

\[
C(U,t,X) = \langle \sigma(v_1,t,X), \ldots, \sigma(v_l,t,X) \rangle
\]

is a vector of the states of the vertices of \( U \) at the beginning of round \( t \) of the execution \( \phi^X \). Denote by \( C[U,t] \) the collection of all possible configurations of the subset \( U \subseteq V \) at time \( t \) over all executions \( \phi^X \) of algorithm \( A_{maid} \) (i.e., on all legal inputs \( X \) ), and let \( \rho[U,t] = |C[U,t]| \).

Prior to the beginning of the execution (i.e., at the beginning of round \( t = 0 \) ), the input string \( X \) is known only to the sender \( s \). The rest of the vertices are found in some initial state, described by the configuration \( C_{init} = C(T_0,0,X) \), which is independent of \( X \). Thus in particular \( \rho[T_0,0] = 1 \). (Note, however, that \( \rho[V,0] = 2^{m^2} \).)

Our main lemma is the following.

**Lemma 3.1.** For every \( 0 \leq t < m^2 \),

\[
\rho[T_{t+1},t+1] \leq (2^{m-1} + 1) \cdot \rho[T_t,t].
\]

**Proof Sketch:** The lemma is proved by showing that in round \( t + 1 \) of the algorithm, each configuration in \( C[T_t,t] \) diverges into at most \( 2^{m-1} + 1 \) different configurations of \( C[T_{t+1},t+1] \).

Fix a configuration \( \hat{C} \in C[T_t,t] \), and let \( \delta = \beta(t + 1) \). The tail set \( T_{t+1} \) is connected to the rest of the graph by the highway edge \( f_1 = (h_{(t+1)m}, h_{(t+1)m}) \) and by the \( m^2 \) path edges \( e_j^1 = (v^j_1, v^j_{1+1}) \), \( 1 \leq j \leq m^2 \). (See Fig. 4.)

Let us count the number of different configurations in \( C[T_{t+1},t+1] \) that may result of the configuration \( \hat{C} \).
Starting from the configuration $\tilde{C}$, each vertex $v^j_i$ is restricted to a single state, and hence it sends a single (well determined) message over the edge $e^j_i$ to $v^j_{i+1}$, thus not introducing any divergence in the execution. The same applies to all the edges internal to $T_{t+1}$. As for the highway edge $f_{i-1}$, the vertex $h_{i-1}^j$ is not in the set $T_t$, hence it may be in any one of many possible states, and the value passed over this edge into the set $T_{t+1}$ is not determined by the configuration $\tilde{C}$. However, due to the restriction of the B-bounded-message model, at most $2^B + 1$ different behaviors of $f_{i-1}$ can be observed by $h_{i+1}$. Thus altogether, the configuration $\tilde{C}$ diverges into at most $2^B + 1$ possible configurations $\tilde{C}_1, \ldots, \tilde{C}_{2^B + 1} \in \mathcal{C}[T_{t+1}, t + 1]$, differing only by the state $\sigma(h_{i+1}, t + 1, X')$. The lemma follows.

Applying Lemma 3.1 and the fact that $\rho[T_0, 0] = 1$, we get the following result. (Some of the proofs are omitted from this extended abstract.)

**Corollary 3.2.** For every $0 \leq t < m^2$, 

$$\rho[T_t, t] \leq (2^B + 1)^t.$$ 

Let $t_{\text{end}}$ denote the time it takes algorithm $A_{\text{mail}}$ to complete the mailing. As argued earlier, at that time, the receiver $r$ may be in at least $2m^2$ different states, hence necessarily $\rho[T_n, t_{\text{end}}] \geq 2m^2$. Applying Cor. 3.2, we get that $(2^B + 1)^{t_{\text{end}}} \geq 2m^2$, implying the following.

**Lemma 3.3.** For every $m \geq 1$, solving the mailing problem $\text{Mail}(F^2_m, h_0, h_{m^2}, m^2)$ in the B model requires $\Omega(m^2/B)$ time.

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### 4 A lower bound for the MST problem on $F^2_m$

In this section we use the results achieved in the previous sections, and show that the distributed MST problem cannot be solved faster than $\Omega(m^2/B)$ on weighted versions of the graphs $F^2_m$.

Formally, the minimum spanning tree (MST) problem can be stated as follows. Given a graph $G(V, E)$ and a weight function $\omega$ on the edges, it is required to find a spanning tree $MST(G) \subseteq E$ whose total weight, $\omega(MST(G)) = \sum_{e \in MST(G)} \omega(e)$, is minimal. In the distributed model, the input and output of the MST problem are represented as follows. Each vertex knows the IDs of its closest neighbors and the weights of the corresponding edges. A degree-$d$ vertex $v \in V$ with neighbors $u_1, \ldots, u_d$ has $d$ weight variables $W^v_{1}, \ldots, W^v_{d}$, with $W^v_{i}$ containing the weight of the edge connecting $v$ to $u_i$, i.e. $W^v_{i} = \omega(v, u_i)$. The output of the MST problem at each vertex $v$ is an assignment to the (boolean) output variables $Y^v_1, \ldots, Y^v_{d}$, assigning

$$Y^v_{i} = \begin{cases} 1, & (u_i, v) \in MST(G), \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove the lower bound, we define in Subsection 4.1 a family of weighted graphs $J^2_m$, based on $F^2_m$ but differing in their weight assignments. Then, in Subsection 4.2, we show that any algorithm solving the MST problem on the graphs of $J^2_m$ can also be used to solve the mailing problem on $F^2_m$ with the same time complexity. Subsequently, the lower bound for the distributed MST problem follows from the lower bound given in the previous section for the mailing problem in $F^2_m$.

#### 4.1 The graph family $J^2_m$

The graphs $F^2_m$ defined earlier were unweighted. In this subsection, we define for every graph $F^2_m$ a family of weighted graphs

$$J^2_m = \left\{ J^2_{m, \omega} = (F^2_m, \omega) \right\}_{1 \leq \gamma \leq 2m^2},$$

where $\omega$ is an edge weight function.

Recall that in the graph $F^2_m$ there are three types of edges, namely, highway edges, edges of paths $P^j$, and star spokes. In all the weight functions $\omega$, all the edges of the highway $H$ and the paths $P^j$ are assigned the weight 0. The spokes of all stars except $S_0$ and $S_m$ are assigned an infinite weight. The spokes of the star $S_0$ are assigned the weight 2.
The only differences between different weight functions \( \omega \), occur on the \( m^2 \) spokes of the star \( S_0 \). Specifically, each of these \( m^2 \) spokes is assigned a weight of either 1 or 3; there are thus \( 2^{m^2} \) possible combinations of weight assignments. (See Fig. 5.)

\[
\begin{array}{c}
S_0 \\
0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \\
1 \quad \\
3: \\
1 \\
\end{array} \\
\begin{array}{c}
S_m \\
0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \\
2 \\
2: \\
2 \\
\end{array}
\]

\[s = h_0, \quad r = h_{m^2}\]

**Figure 5:** The edge weights assigned to \( J_{m, \gamma}^2 \).

Since discarding any combination of infinite weight edges from the graph \( J_{m, \gamma}^2 \) leaves it connected, and since any tree containing an infinite weight edge has infinite weight, the following is clear.

**Lemma 4.1.** No infinite weight spoke edge belongs to the MST of \( J_{m, \gamma}^2 \), for every \( 1 \leq \gamma \leq 2m^2 \).

Let us pair the spoke edges of \( S_0 \) and \( S_m \), denoting the \( j \)th pair (for \( 1 \leq j \leq m^2 \)) by

\[PE^j = \{(s, v_0^j), (r, v_{m^2}^j)\} .\]

**Lemma 4.2.** For every \( 1 \leq \gamma \leq 2m^2 \) and \( 1 \leq j \leq m^2 \), exactly one of the two edges of \( PE^j \) belongs to the MST of \( J_{m, \gamma}^2 \), namely, the lighter one.

**Proof:** Since the MST must be connected, at least one of the two edges of \( PE^j \) must belong to it, as otherwise the path \( P^j \) is completely disconnected from the rest of the graph, by Lemma 4.1. It remains to show that the MST cannot contain both edges of \( PE^j \).

The proof is by contradiction. Consider the cycle in \( J_{m, \gamma}^2 \), consisting of the edges of \( H \), \( PE^j \) and \( P^j \), and suppose that both edges of \( PE^j \) are in the MST. In order for the MST to be cycle-free, at least one edge \( e \) of either the highway \( H \) or the path \( P^j \) must not belong to the MST. Since the edges of \( H \) and \( P^j \) have zero weight, \( \omega^j(e) = 0 \). Hence deleting the heavier edge of the pair \( PE^j \) and adding the edge \( e \) instead leaves us with a lighter tree than the original one, leading us to contradiction. \( \square \)

**Lemma 4.3.** For every \( m \geq 2 \) and \( 1 \leq \gamma \leq 2m^2 \), all the edges of the highway \( H \) and the paths \( P^j \), for \( 1 \leq j \leq m^2 \), belong to the MST of \( J_{m, \gamma}^2 \).

Fig. 6 illustrates the remaining candidate edges to join the MST. Bold edges belong to the MST under any edge weight function. Of the remaining edges, exactly one of each pair will join the MST, depending on the particular weight assignment to the spoke edges of the star \( S_0 \).

\[
\begin{array}{c}
S_0 \\
0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \\
1 \quad \\
3: \\
1 \\
\end{array} \\
\begin{array}{c}
S_m \\
0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \\
2 \\
2: \\
2 \\
\end{array}
\]

\[s = h_0, \quad r = h_{m^2}\]

**Figure 6:** The remaining candidate edges to join the MST of \( J_{m, \gamma}^2 \): Bold edges belong to the MST under any edge weight function.

### 4.2 A lower bound on MST in \( J_{m}^2 \)

**Lemma 4.4.** Any distributed algorithm for constructing an MST on the graphs of the class \( J_{m}^2 \), can be used to solve the Naïl \( (F_m^2, h_0, h_{m^2}, m^2) \) problem on \( F_m^2 \) with the same time complexity.

**Proof:** Consider an algorithm \( A_{mst} \) for the MST problem, and suppose that we are given an instance of the Naïl \( (F_m^2, h_0, h_{m^2}, m^2) \) problem with input string \( X \). We use the algorithm \( A_{mst} \) to solve this instance of the mailing problem as follows. The sender \( s = h_0 \) initiates the construction of an instance of the MST by turning \( F_m^2 \) into a weighted graph from \( J_{m}^2 \), setting the edge weights as follows: for each \( x_i \in X, 1 \leq i \leq m^2 \), it sets the weight variable \( W_{i}^s \) corresponding to the spoke edge \( e_i \in E(S_0) \), to be

\[W_{i}^s = \begin{cases} 3, & \text{if } x_i = 1; \\ 1, & \text{if } x_i = 0. \end{cases}\]

The rest of the graph edges are assigned fixed weights as specified in Subsection 4.1. Note that the weights
for all the edges except those connecting \( s \) to its immediate neighbors in \( S_0 \) do not depend on the particular input instance at hand. Hence a single round of communication between \( s \) and its \( S_0 \) neighbors suffices for performing this assignment; \( s \) assigns its edges weights according to its input string \( \lambda \), and needs to send at most one message to each of its neighbors on \( S_0 \), to notify it about the weight of the spoke connecting them.

Every vertex \( v \) in the network, upon receiving the first message of algorithm \( A_{\text{mst}} \), assigns the values defined by the edge weight function \( \omega_\gamma \) to its variables \( W^v_\gamma \). (As discussed earlier, this does not require \( v \) to know \( \gamma \), as its assignment is identical under all weight functions \( \omega_\gamma \), \( 1 \leq \gamma \leq 2^m \)). From this point on, \( v \) may proceed with executing algorithm \( A_{\text{mst}} \) for the MST problem.

Once algorithm \( A_{\text{mst}} \) terminates, the receiver vertex \( r \) determines its output for the mailing problem, by setting \( X^r_\gamma \leftarrow Y^r_\gamma \) for \( 1 \leq i \leq m^2 \).

By Lemma 4.2, the lighter edge of each pair \( PE^j \), for \( 1 \leq j \leq m \), belongs to the MST; thus, in the set of variables \( Y^r_1, \ldots, Y^r_{m^2} \) obtained by the vertex \( r \) as a result of solving the MST problem, \( Y^r_j \) corresponds to the assignment of \( \omega(h_0, v^r_i) = 3 \) to the \( j \)th edge of \( S_0 \), while \( Y^r_j = 0 \) corresponds to the assignment of 1 to that edge, hence for every \( j \), \( Y^r_j \) equals \( x_j \), the \( j \)th bit of \( \lambda \). Hence the resulting algorithm has correctly solved the given instance of the mailing problem. \( \Box \)

**Theorem 4.5.** For every \( m \geq 1 \), any distributed algorithm for constructing an MST on the graphs of the family \( \mathcal{J}_m^2 \) in the B model requires \( \Omega(m^2/B) \) time.

**Corollary 4.6.** Any distributed algorithm for the MST problem in the B model requires \( \Omega(\sqrt{n}/B) \) time on some \( n \)-vertex graphs of diameter \( O(n^{1/3}) \).

## 5 Lower bounds on \( F^K_m \)

This section generalizes the results of the previous section to the graphs \( F^K_m \) for \( K \geq 3 \), thus establishing the desired lower bound.

### 5.1 The graphs \( F^K_m \)

Given two integer parameters \( m, K \geq 2 \), construct the graph \( F^K_m \) as follows. The two basic units are still the path and the highway, with the following changes. The basic path \( P \) now has \( m^K + 1 \) vertices, i.e.,

\[
V(P) = \{ v_0, \ldots, v_{m^K} \}
\]

and

\[
E(P) = \{ e_i = (v_i, v_{i+1}) \mid 0 \leq i \leq m^K - 1 \}.
\]

There are \( K - 1 \) highways, denoted \( H^1, \ldots, H^{K-1} \). The level-\( \ell \) highway \( H^\ell \) consists of \( m^\ell + 1 \) vertices, i.e.,

\[
V(H^\ell) = \{ h^\ell_{im^K} \mid 0 \leq i \leq m^\ell \}
\]

and

\[
E(H^\ell) = \{ f^\ell_i = (h^\ell_{im^K}, h^\ell_{i+1,m^K}) \mid 0 \leq i \leq m^\ell - 1 \}.
\]

Each highway vertex \( h^\ell_i \) is connected to the corresponding path vertex \( v^\ell_i \) by a spoke edge. Fig. 7 depicts these connections for the case of \( m = K = 3 \).

The graph \( F^K_m \) is constructed by taking \( m^K \) copies of the path \( P \), denoted \( P_1, \ldots, P_{m^K} \), and connecting them all to the same level-\( \ell \) highway \( H^\ell \), for each \( 1 \leq \ell \leq K - 1 \). The vertex \( h^\ell_m \) is the intended sender \( s \), and the vertex \( h^\ell_{m^K} \) is the intended receiver \( r \). (See Fig. 8, showing the graph \( F^3_3 \).)

**Fact 5.1.** The cardinality of \( F^K_m \) is \( n = O(m^{2K}) \) and its diameter is \( O(Km) \).

### 5.2 Generalizing the lower bound on mailing to the graphs \( F^K_m \)

The lower bound for the mailing problem can be extended from \( F^2_m \) to \( F^K_m \) for \( K \geq 3 \) in a natural way, sketched next. Consider some arbitrary algorithm \( A_{\text{mst}} \), and let \( \varphi_{\lambda} \) denote the execution of \( A_{\text{mst}} \) on the input \( \lambda \) in the graph \( F^K_m \). The notion of a tail set is generalized to \( F^K_m \) for \( K \geq 3 \) as follows. For every \( 1 \leq j \leq m \), define the tail of the path \( P^j \) as before, i.e.,

\[
T_i(P^j) = \{ v^j_i \mid i \leq l \leq m^K \}.
\]

Let \( \beta^j(i) \) denote the least integer \( \delta \) such that \( \delta m^\ell \geq i \), and define the tail of \( H^\ell \) as

\[
T_i(H^\ell) = \{ h^\ell_{j m^K} \mid \beta^j(i) \leq j \leq m^\ell \}.
\]

Now the tail set of \( F^K_m \) becomes the union of those tails,

\[
T_i = T_i(H^\ell) \cup \bigcup_j T_i(P^j).
\]

(See Fig. 8.) Again, for \( i = 0 \) the definition is \( T_0 = V \setminus \{ h^\ell_{m^K} \} \). The main lemma becomes the natural extension of Lemma 3.1, and its proof is similar.

**Lemma 5.1.** For any \( 0 \leq t < m^K \),

\[
\rho(T_i, t+1) \leq (2^B + 1)^{K-1} \cdot \rho(T_i, t) .
\]
Corollary 5.2. For any $0 \leq t < m^K$,

$$\rho[T_t, t] \leq (2^h + 1)^{K_1}t.$$

Letting $t_{end}$ denote the time it takes algorithm $A_{mis}$ to complete the mailing, we derive, similar to the proof for $K = 2$, that necessarily

$$(2^h + 1)^{K_1}t_{end} \geq \rho[t_{end}, t_{end}] \geq 2^{m^K},$$

implying the following.

Lemma 5.3. For every $K, m \geq 2$, solving the mailing problem $\text{Mail}(F^K_m, h^1_m, h^2_m, m^K)$ in the $B$ model requires $\Omega(m^K/BK)$ time.

5.3 Generalizing the lower bound on MST to $\mathcal{F}^K_m$

Finally we sketch the proof of the lower bound for the MST problem on the weighted versions of the graphs $F^K_m$. Define the families of weighted graphs $\mathcal{J}^K_m$ as follows. For every two integers $m, K \geq 2$, let

$$\mathcal{J}^K_m = \left\{ J^K_{m, \gamma} = (F^K_m, \omega^K_{\gamma}) \mid 1 \leq \gamma \leq 2^{m^K} \right\},$$

where $\omega^K_{\gamma}$ is the weight function defined as follows. All the edges of the highways $H^\ell$ for $1 \leq \ell \leq K - 1$ and the paths $P^j$ for $1 \leq j \leq m^K$ are assigned zero weight. Group all the spoke edges connecting the vertices $h^\ell_j$ of the highways to the vertices $v^j_i$ of the paths into a multicentered star $S^K$, thus partitioning the collection of spoke edges into the multicentered stars $S^K_0, \ldots, S^K_{m^K}$. The spoke edges of all multicentered stars except for the two extreme ones, $S^K_0$ and $S^K_{m^K}$, are assigned infinite weight. It remains to describe the weight assignment for the edges of the remaining two stars.

The spoke edges of $S^K_{m^K}$ that connect the endpoints of highways $H^\ell$ for $2 \leq \ell \leq K - 1$ to the paths $P^2, \ldots, P^{m^K}$ are assigned infinite weight. The spoke edges connecting $H^\ell$ for $2 \leq \ell \leq K - 1$ to the endpoint $v^j_1$ of $P^j$ are assigned zero weight.

The spoke edges connecting the endpoint of $H^1$ to the endpoints of the paths $P^1, \ldots, P^{m^K}$ are assigned weight 2.

The spoke edges of $S^K_0$ connecting the end vertices of $H^\ell$ for $2 \leq \ell \leq K - 1$ to the endpoints of the paths $P^1, \ldots, P^{m^K}$ are assigned infinite weight. The spoke edges of $S^K_0$ connecting the highway $H^1$ to the end vertices of the paths $P^1, \ldots, P^{m^K}$ are assigned weight 1 or 3, depending on $\gamma$, as in Subsection 4.1.

The following is clear from the previous discussion.

Lemma 5.4. No infinite weight spoke edge belongs to the MST of $J^K_{m, \gamma}$, for every $1 \leq \gamma \leq m^K$.

Since the highways $H^\ell$ for $2 \leq \ell \leq K - 1$ need to be connected to the rest of the graph, and the only finite weight edge connected to them is the spoke of the star $S^K_{m^K}$ connecting it to $P^1$, we conclude the following.

Lemma 5.5. For every $2 \leq \ell \leq K - 1$, the edge $(h^\ell_1, v^j_1)$ belongs to the MST.

Let us pair the spoke edges of $S^K_0$ and $S^K_{m^K}$ connecting $H^1$ to $P^j$ for $1 \leq j \leq m^K$, denoting the $j$th pair (for $1 \leq j \leq m^K$) as

$$PE^j = \left\{ (s, v^j_0), (r, v^j_{m^K}) \right\}.$$

Lemma 5.6. For every $1 \leq j \leq m^K$, exactly one of the two edges of $PE^j$ belongs to the MST of $J^K_{m, \gamma}$, namely, the lighter one.

We obtained an instance of MST problem, in which the membership of edges in the MST is predetermined for all but the $m^K$ edge pairs $PE^j$. Following the proof method of Lemma 4.4, we show that
any algorithm solving the distributed MST problem on $\mathcal{J}_m^K$ can be used for solving the mailing problem in the same time complexity, implying the following.

**Theorem 5.7.** For every $m,K \geq 2$, any distributed algorithm for constructing an MST on the graphs of the family $\mathcal{J}_m^K$ in the B model requires $\Omega(m^K/BK)$ time.

**Corollary 5.8.** For every $K \geq 2$, there exists a family of $n$-vertex graphs of diameter $O(Kn^{1/2K})$, such that any distributed algorithm for the MST problem in the B model requires $\Omega(\sqrt{n}/BK)$ time on some of those graphs.

**Corollary 5.9.** For every $n \geq 2$, there exists a family of $n$-vertex graphs of diameter $O(\log n)$ such that any distributed algorithm for the MST problem in the B model requires $\Omega(\sqrt{n}/B\log n)$ time on some of those graphs.

**References**


