Approximation Algorithms for Art Gallery Problems in Polygons

Subir K. Ghosh\(^1\) ghosh@tifr.res.in

\(^1\)School of Technology & Computer Science, Tata Institute of Fundamental Research
Mumbai 400005, India
http://www.tcs.tifr.res.in/~ghosh/

The art gallery problem is to determine the number of guards that are sufficient to cover or see every point in the interior of an art gallery. An art gallery can be viewed as a polygon \(P\) with or without holes with a total of \(n\) vertices and guards as points in \(P\). Any point \(z \in P\) is said to be visible from a guard \(g\) if the line segment joining \(z\) and \(g\) does not intersect the exterior of \(P\). Usually the guards may be placed anywhere inside \(P\). If the guards are restricted to vertices of \(P\), we call them vertex guards. If there is no restriction, the guards are referred as point guards. Point and vertex guards are also referred as stationary guards. If the guards are mobile, i.e., able to patrol along a segment inside \(P\), they are called mobile guards. If the mobile guards are restricted to edges of \(P\), they are called edge guards.

The art gallery problem was first posed by Victor Klee for stationary guards. Chavatal [6] proved that a simple polygon \(P\) needs at most \(\lfloor n/3 \rfloor\) stationary guards. Fisk [11] later gave a simple proof of this result using coloring technique, and based on his proof, Avis and Toussaint [2] developed an \(O(n \log n)\) time algorithm for positioning guards in \(P\). O’Rourke [22] showed that \(P\) needs at most \(\lfloor n/4 \rfloor\) mobile guards. For edge guards, \(\lfloor n/4 \rfloor\) edge guards seem to be sufficient for guarding \(P\), except for a few polygons (see [27]).

For a simple orthogonal polygon \(P\), i.e., the edges of \(P\) are horizontal or vertical, Kahn et al. [16] proved that \(P\) needs at most \(\lfloor n/4 \rfloor\) stationary guards. O’Rourke [21] later gave an alternative proof for this result. These proofs use the partition of \(P\) into convex quadrilaterals before \(\lfloor n/4 \rfloor\) guards are placed in \(P\). Note that a convex quadrilaterization of \(P\) can be obtained by algorithms of Edelsbrunner, O’Rourke and Welzl [8], Lubiw [20], Sack [23], and Sack and Toussaint [24]. Aggarwal [1] showed that \(P\) needs at most \(\lfloor 3n+4 \rceil /16\) mobile guards. This bound also holds for edge guards as shown by Bjorling-Sachs [4].

For a polygon \(P\) with \(h\) holes, O’Rourke [22] showed that \(P\) needs at most \(\lfloor n+2h \rceil/3\) vertex guards. Hoffmann et al. [15] and Bjorling-Sachs and Souvaine [5] proved independently that \(P\) can always be guarded with \(\lceil n+h \rceil/3\) point guards. Bjorling-Sachs and Souvaine also gave an \(O(n^2)\) time algorithm for positioning the guards.
There is no tight bound known on the number of mobile guards required to guard $P$. Since $\lceil \frac{n+h}{3} \rceil$ point guards are sufficient to guard $P$, the bound naturally holds for mobile guards as well. To guard an orthogonal polygon $P$ with $h$ holes, Győri et al. [14] proved that $\lceil \frac{3n+4h+1}{16} \rceil$ mobile guards are always sufficient to guard $P$. For survey of art gallery theorems and algorithms, see Ghosh [13], O’Rourke [22], Shermer [26] and Urrutia [27].

The minimum guard problem is to find the minimum number of guards that can see every internal point of a polygon. O’Rourke and Supowit [22] showed that the minimum vertex, point and edge guard problems in polygons with holes are NP-hard. Even in the case of polygons without holes, Lee and Lin [19] showed that the minimum vertex, point and edge guard problems are NP-hard. The minimum vertex and point guard problems are also NP-hard for simple orthogonal polygons as shown by Katz and Rpoisman [17] and Schuchardt and H.-D. Hecker [25].

Ghosh [12] presented approximation algorithms for minimum vertex and edge guard problems for polygons with or without holes by transforming art gallery problems into set-cover problems. For simple polygons $P$, approximation algorithms for both problems run in $O(n^4)$ time (after a recent improvement) and yield solutions that can be at most $O(\log n)$ times the optimal solution. For polygons $P$ with holes, approximation algorithms for both problems give the same approximation ratio of $O(\log n)$ but the algorithms take $O(n^5)$ time (after a recent improvement). For the last two decades, this is the only known technique for transforming these four art gallery problems leading to efficient approximation algorithms in terms of worst case running times and approximation bounds.

Regarding the lower bound on the approximation ratio for the problems of minimum vertex, point and edge guards in simple polygons, Eidenbenz, Stamm and Widmayer [10] showed that these problems are APX-hard. This means that for each of these problems, there exists a constant $\epsilon > 0$ such that an approximation ratio of $1 + \epsilon$ cannot be guaranteed by any polynomial time approximation algorithm unless $P = NP$. Though there may be approximation algorithms for these problems whose approximation ratios are not small constants, for polygons with holes, these problems cannot be approximated by a polynomial time algorithm with ratio $((1-\epsilon)/12)(\ln n)$ for any $\epsilon > 0$, unless $NP \subseteq TIME(n^{O(\log \log n)})$. The results are obtained by using gap-preserving reductions from the SET COVER problem. So, the open problem is to design approximation algorithms for vertex, edge and point guards problems in simple polygons which yield solutions within a constant factor of the optimal.

Recently, Efrat and Har-Peled [9] presented randomized approximation algorithms for the minimum vertex guard problem in polygons. For simple polygons $P$, the randomized approximation algorithm runs in $O(n c_{opt}^2 \log^4 n)$ expected time and the
approximation ratio is $O(\log c_{opt})$, where $c_{opt}$ is the number of vertices in the optimal solution. In the worst case, $c_{opt}$ can be a fraction of $n$. For polygons $P$ with $h$ holes, the randomized approximation algorithm runs in $O(nhc_{opt}^3 \text{polylog } n)$ expected time and the approximation ratio is $O(\log n \log(c_{opt} \log n))$. Note that their randomized approximation algorithms do not always guarantee solutions and the quality of approximation is correct with high probability. No other approximation algorithm (deterministic or randomized) is known for the minimum vertex or edge guard problem in polygons. For special classes of polygons, there are approximation algorithms for the minimum point guard problem. Also, there are approximation algorithms (i) for the minimum vertex and point guard problems in 1.5-dimensional terrains, and (ii) for the minimum vertex guard problem in 2.5-dimensional terrains. [3, 7, 17, 18].

References


